

*The Taylor Principle and the Taylor Rule Determinacy
Condition in the Baseline New Keynesian Model: Two Different
Kettles of Fish*

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Abstract

The Taylor Principle (1993) suggests monetary policy should make the interest rate move in the same direction and by a greater amount than observed movements in inflation. The resulting co-movements between inflation and the real interest rate, as Taylor (1999) demonstrated for a simple, “backward-looking” model, can be shown to be necessary for stability in models of that type. In the context of the baseline New Keynesian, “forward-looking” model, (NKM) as expounded by Woodford (2001, 2003) a related “stability and uniqueness” (or determinacy) condition arises which is necessary for a determinate solution. Woodford and many others have interpreted this condition as representing the Taylor Principle condition.

In this paper we argue that (1) the dynamic interpretation Woodford and others put on the NKM determinacy condition is inappropriate; (2) the standard NKM determinacy condition does not even require (in general) the Taylor Principle (i.e., real interest rate moving together with inflation) to hold. The Taylor Principle and the NKM determinacy condition are two different kettles of fish.

I. Introduction

In recent years a great deal of attention among monetary economists has been focused on the issue of uniqueness, stability and/or “determinacy” in macroeconomic models, particular in the New Keynesian Model (NKM). Modeling in NKM usually represents monetary policy as following a Taylor rule, and the parameters of the Taylor rule must meet certain “determinacy conditions” which may be necessary to rule out both “sunspots” and explosive solutions. At least since the widely-cited article by Woodford (2001) there has been a consensus that the determinacy condition in these models is essentially a restatement of the Taylor Principle—that the policy rule must guarantee that *real interest rates will move in the same direction as inflation*. This movement of real interest rates contains demand and inflation responses to shocks that would otherwise create explosive or stable sunspot solutions. We beg to differ with this line of reasoning, at least as applied to the NKM.

Our argument is as follows. “Backward-looking” models (such as that by Taylor (1999) himself) do require the Taylor Principle to hold for “stability”—meaning here where projected paths eventually approach long-run or steady-state solutions of the endogenous variables. But one should avoid conflating the Taylor Principle with the determinacy condition in the forward-looking NKM. The determinacy condition in NKM, when met, assures us that the model’s solution is unique. The dynamics are “stable” in the above sense by construction (all shocks are AR(1)). If the determinacy condition is not met, the solution is “immediately explosive” (no finite solution exists) or may result in non-explosive sunspot solutions.¹

To illustrate our point, we begin by showing how the two concepts apply in simple, univariate backward- and forward-looking models. Next we turn to two representative bivariate (inflation and output gap) backward- and forward-looking models. The backward-looking model

is Taylor's (1999) model; the forward-looking model is the baseline NKM as expounded by Clarida, Gali and Gertler (1999), Woodford (2001, 2003) and others. As a kind of *coup de grace* to the proposition that determinacy means Taylor Principle, we produce two examples of solutions where the determinacy condition is met in the baseline NKM with Taylor rule, but in which real interest rates move in the opposite direction of inflation.

II. A Univariate Backward-Looking Model

A basic univariate backward-looking model can be written:

$$(1) \quad \tilde{x}_t = b\tilde{x}_{t-1} + u_t,$$

where $b > 0$, $u_t = \rho u_{t-1} + \eta_t$ and $0 < \rho < 1$. η_t is a white noise. Iterating the model forward, we rewrite (1) as:

$$(2) \quad \tilde{x}_{t+n} = b^{n+1}\tilde{x}_{t-1} + \sum_{i=0}^n \rho^i b^{n-i} u_t,$$

where n is the number of future periods. From equation (2) we see that dynamics of u_t process determine the future path of \tilde{x}_t . Figure 1 illustrates the effects of a positive unit shock of u_t on \tilde{x}_t . These paths involve well-defined equilibria for different times. We consider only positive values of b . The value of b does determine whether the path is a convergent path or a divergent path. Whether b is greater or less than one, the solutions are determinate. As $n \rightarrow \infty$ with $b < 1$, the solution path exists and it converges to a certain value.² As $n \rightarrow \infty$ with $b \geq 1$, the solution path approaches infinity ($b > 1$) or settles away from the long-run equilibrium of the model ($b = 1$). This is "instability" in the backward-looking case.

III. A Univariate Forward-Looking Model

Contrast the above with a forward-looking univariate model such as

$$(3) \quad \vec{x}_t = bE_t[\vec{x}_{t+1}] + u_t,$$

where $b > 0$, $u_t = \rho u_{t-1} + \eta_t$ and $0 < \rho < 1$. η_t is a white noise. Again using the iteration method, we rewrite equation (3) as:

$$(4) \quad \vec{x}_t = b^n E_t[\vec{x}_{t+n}] + (1 + b\rho + b^2\rho^2 + \dots + b^n\rho^n)u_t.^3$$

The necessary condition for convergence in equation (4) is $b < 1$ which guarantees that the second term of the right hand side of equation (4) has a defined limit, and the limit of the first term of the right hand side of equation (4) is zero. If sunspots are present in the structure of $E_t[\vec{x}_{t+n}]$, they will drop out of the solution. The value of solution paths will follow

$$(5) \quad \vec{x}_t = \frac{1}{1-b\rho} u_t,$$

and convergence to this path is immediate.⁴ If $b \geq 1$ and there are no sunspots, the model explodes instantaneously ($b > 1$) or indeterminate ($b = 1$). If $b \geq 1$ and there are sunspots, there may be non-explosive but arbitrary solutions. Either way, we say the solution is “indeterminate.”

It should be noted that different values of $b (< 1)$ in equation (5) imply different solution paths. Figure 2 illustrates the dynamics for different paths upon a unit shock of u_t on \vec{x}_t . Where a determinate solution exists, the path is stable (tendency to return to the long-run solution) by construction.

IV. Taylor (1999)

Now consider the bivariate (inflation and output gap) counterparts to univariate models.

We start with the bivariate backward-looking model by Taylor (1999):

$$(6) \quad x_t = -\beta(i_t - \pi_t - r) + g_t$$

$$(7) \quad \pi_t = \pi_{t-1} + \alpha x_{t-1} + u_t$$

$$(8) \quad i_t = \varphi_0 + \varphi_\pi \pi_t + \varphi_x x_t$$

where x represents the output gap in logs; π the inflation rate; i the short term nominal interest rate. g and u are (independent and not auto-correlated) shocks with zero mean. The model parameters α and β are positive. r is the natural rate of interest. φ_0 , φ_π and φ_x are the policy parameters of the Taylor rule.

The solution path of inflation can be written as:

$$(9) \quad \pi_{t+i} = \sum_{i=1}^n \Lambda^{i-1} g_{t+i} + \frac{\alpha}{1+\beta\varphi_x} \sum_{i=1}^n \Lambda^{i-1} u_{t+i} + \Lambda^{i-1} \pi_t + \alpha \Lambda^{i-1} x_t$$

where $\Lambda = \frac{\alpha\beta(1-\varphi_\pi)+(1+\beta\varphi_x)}{1+\beta\varphi_x}$. The key parameter is Λ , which if inside the unit circle will mean

that projections of π_{t+1} will eventually approach the model's steady-state value $(\frac{\varphi_0-r}{1-\varphi_\pi})$ —hence,

the model is “stable” in this sense Taylor intended. If Λ lies outside the unit circle the model's

projections will depart continuously from steady-state values; if Λ lies on the unit circle the

model's projection will approach a value that differs from the steady-state value. In these cases

the model is “unstable” in the Taylor sense, though not “explosive” in the immediate sense.

Each period's projection is well-defined.

Since $\Lambda = 1 + \frac{\alpha\beta(1-\varphi_\pi)}{1+\beta\varphi_x}$ the Taylor stability condition can be simplified to

$1 < \varphi_\pi < 1 + \frac{2(1+\beta\varphi_x)}{\alpha\beta}$. This establishes the Taylor Principle and its role in this model.

Taylor (1999) and Cochrane (2011) in his description of the Taylor (1999) model neglect the upper bound required on φ_π , emphasizing only the requirement that $\varphi_\pi > 1$ for Taylor stability.⁵

V. Clarida, Gali and Gertler (1999)

Now while the Taylor Principle governs whether a backward-looking model like Taylor's above will be "stable" in terms of tending toward steady-state, there is no similar implication for forward-looking models. To illustrate this point, we turn to the bivariate forward-looking model of Clarida, Gali and Gertler (1999, henceforth CGG (1999)). Their baseline NKM consists of the familiar equations for the output gap ("IS") and inflation ("Phillips curve"):

$$(10) \quad x_t = E_t[x_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}]) + g_t$$

$$(11) \quad \pi_t = kx_t + \beta E_t[\pi_{t+1}] + u_t$$

where x again represents output gap in logs; π inflation (log-deviation from steady-state); and i represents the nominal interest rate (deviation from steady-state); $0 < \beta < 1$ is a discount factor; $k > 0$ is the Phillips curve parameter reflecting the degree of price flexibility (higher means more), and $\sigma > 0$ is the consumption-elasticity of utility. g_t and u_t are shocks of AR(1) form: $u_t = \rho u_{t-1} + \eta_t$ ($0 < \rho < 1$) and $g_t = \lambda g_{t-1} + \varepsilon_t$ ($0 < \lambda < 1$).

The solutions for the optimal path of CGG (1999) under discretion (time-consistent) as:

$$(12) \quad x_t = -\frac{k}{k^2 + \Gamma(1 - \beta\rho)} u_t$$

$$(13) \quad \pi_t = \frac{\Gamma}{k^2 + \Gamma(1 - \beta\rho)} u_t$$

where Γ is the weight on the output gap in the welfare function.

Three observations should be made about these paths: first, the IS or prospective productivity shock g_t does not appear. Since these “demand-shocks” move π_t and x_t in the same direction, the optimal path requires there be no response on the part of these variables to g_t shocks. Second, these paths are by construction “stable” (i.e., in the sense the variables approach steady state solutions) on account of the specification of the u_t shocks. Third, these paths can be (were) derived without specification of any particular monetary policy rule. To learn whether paths (12) and (13) will be obtained it is essential to specify the monetary policy rule and its determinacy conditions.

VI. Optimality, Determinacy and the Taylor Rule

A variety of monetary policy rules can meet the requirements of (12) and (13), but the Taylor rule has been almost universally employed. We write the particular Taylor rule as

$$(14) \quad i_t = \varphi_\pi \pi_t + \varphi_x x_t + \varphi_g g_t$$

which includes no intercept (the model is derived as log-linearized deviations from steady-state)

and contains a φ_g which can be used to offset g_t shocks.⁶ Putting (14) into (10) and (11), the

model can be written in the matrix form $\mathbf{M}_t = \mathbf{A}\mathbf{M}_{t+1} + \mathbf{e}_t$, where $\mathbf{M}'_t = [x_t \ \pi_t]$, $\mathbf{M}'_{t+1} =$

$[E_t[x_{t+1}] \ E_t[\pi_{t+1}]]$, \mathbf{A} is the two-by-two coefficient matrix and \mathbf{e}_t is a vector of exogenous shock

terms. The solution of the model will be linear in u_t and g_t . Bullard and Mitra (2002) showed that determinacy of this model requires that the eigenvalues of \mathbf{A} lie within the unit circle, which in turn requires

$$(15) \quad k(\varphi_\pi - 1) + (1 - \beta)\varphi_x > 0.$$

This is the condition that rules out explosive solutions as well as “sunspot” solutions.

In this case the optimal Taylor rule, as derived by Thurston (2012) for the model (10), (11), and (14) is

$$(16) \quad \varphi_g = \sigma$$

$$(17) \quad \varphi_\pi = \rho + \frac{k\sigma(1-\rho)}{\Gamma} + \frac{k}{\Gamma}\varphi_x$$

which obviously provides multiple possible values for φ_x and φ_π . As illustrated in Figure 3, the unique, optimal paths (12) and (13) will necessarily be reached for all combinations of φ_x and φ_π that lie to the Northeast of the borderline for determinacy (15).

An interesting implication of the optimal paths created by this optimal Taylor rule—(16) and (17)—is that, since all periods require $x = -\frac{k}{\Gamma}\pi$ ⁷ the model can be written in the form of two, univariate equations (x_t as a function of $E_t[x_{t+1}]$ and π_t as a function of $E_t[\pi_{t+1}]$) along the lines of our univariate, forward-looking example in Section III. In this “optimized” solution paths, the $b < 1$ condition noted in the earlier example again guarantees determinacy, although that the condition turns out to be a more “stringent” condition than is actually required by (15). The point on the optimal $\{\varphi_\pi, \varphi_x\}$ locus in Figure 3 is marked as φ'' . More details can be found in Appendix A.

VII. Determinate Solutions Without the Taylor Principle

With the background and argument of the previous sections we now return to the classic interpretation of the NKM-Taylor rule determinacy condition by Woodford (2001, p. 233), probably the most elegantly expressed and most widely cited:

The determinacy condition...has a simple interpretation. A feedback rule satisfies the Taylor Principle if it implies that in the event of a sustained increase in the inflation rate by k percent, the nominal interest rate will eventually be raised by more than k percent. In the context of the model sketched above, each percentage point of permanent increase in the inflation rate implies an increase in the long-run average output gap of $\frac{1-\beta}{k}$ percent; thus a rule of the form conforms to the Taylor Principle if and only if the coefficients φ_π and φ_x satisfy $\varphi_\pi + \frac{1-\beta}{k}\varphi_x > 1$. In particular, the coefficient values necessarily satisfy the criterion, regardless of the size of β and k . Thus the kind of feedback prescribed in the Taylor rule [$\varphi_x = 0.5, \varphi_\pi = 1.5$] suffices to determine an equilibrium price level.

What is our objection to the statement above? First, the paragraph implies that the determinacy condition influences the dynamic paths of π_t (and presumably x_t), whereas we noted earlier that (15) ensures unique paths for π and x which are reached instantly according to NKM. Second, Woodford's statement suggests that in order for these unique paths to be obtained the Taylor Principle must be applied—i.e., *real interest rates must rise with inflation*. It has become nearly universal to assert that positive co-variation between the real interest rate and inflation is critical in avoiding explosive solutions and sunspots. Appendix C lists recent articles and is grouped according to the level of explicitness with which they conflate the Taylor Principle (real interest rates rising in inflation) with the standard determinacy condition (15).

It turns out that the assertion that the real interest rate must move with inflations in order to establish determinacy is fairly easy to refute. Simply put, the standard NKM determinacy condition (15) does *not require* the real interest rate to co-vary positively with inflation. We

provide two examples: first, where the shock comes from the supply side (u), second from the “IS” side (g).

An example for the u -shock

For this case, we can assume the g shock is continuously equal to 0, or posit that the authorities set the Taylor rule parameter $\varphi_g = \sigma$, which essentially nullifies these demand shocks. Having only one shock means the solution for π_t and x_t will be proportional to u_t , which means that the expected $t + 1$ values of these variables can be written as ρ times their current values. Then taking the total differentials of (10) with i_t represented by the Taylor rule and (11), one arrives as a constraint between movements in the real interest rate r_t and $E_t[\pi_{t+1}]$ as

$$(18) \quad r_t = \left\{ \frac{\sigma(1-\rho)}{(1-\rho)\sigma + \varphi_x} (\varphi_\pi - \rho) \right\} E_t[\pi_{t+1}].$$

Details can be found in Appendix B. The term in the brackets of (18) must be negative if $\varphi_\pi < \rho$ for any positive φ_x , and $\varphi_x > 0$ is required by the determinacy condition (15). For any policy setting that is not explosive or accommodative of sunspots, the real interest rate must *decline* in expected inflation when $\varphi_\pi < \rho$, *rise* in expected inflation when $\varphi_\pi > \rho$ or indeed not change in expected inflation (interest rate change matches inflation) when $\varphi_\pi = \rho$. Figure 4 presents an illustrative simulation of the period t effects of u_t on the real interest rate for various settings of φ_π .⁸ In addition, Figure 5 shows the impulse responses to the unit of u_t in DYNARE/MATLAB with the parameter values of $\rho = 0.5$, $\varphi_\pi = 0.3$, $\varphi_x = 22$, $\varphi_g = \sigma = 1$, $k = 0.3$, $\beta = 0.99$. The effect of u_t is to change r_t in the opposite direction.

It should be remarked that setting $\varphi_\pi < \rho$ (with φ_x high enough to assure determinacy) will never provide an optimal outcome in this baseline NKM. Figure 6 illustrates the effect of

constraining φ_π to be equal or below ρ . Determinate Taylor rule settings are restricted to a range that cannot include the optimal setting.

It is worth noting that the *optimal* setting of the Taylor rule must involve the Taylor Principle (real interest rates rising in inflation), which conforms to the result of CGG (1999, p. 1672) for this model in which the optimal interest rate is (using our notation)

$$(19) \quad i_t = \left\{1 + \frac{(1-\rho)k\sigma}{\rho\Gamma}\right\} E_t[\pi_{t+1}] = \left\{1 + \frac{(1-\rho)k\sigma}{\rho\Gamma}\right\} \rho\pi_t$$

where the term in brackets must be greater than unity. To say that the Taylor Principle *should* apply is of course using “should” in the normative sense.

An example for the g-shock

What if the origin of the inflation in question comes from the demand side (through g)? Does the determinacy condition (15) mean the real interest rate must move in the same direction as inflation? The answer is no, real interest rates can move in the opposite direction as inflation, although the conditions are less straightforward than in the previous example. We have derived two boundary conditions relating φ_π and φ_x in addition to one shown earlier for determinacy. Details also can be found in Appendix B.

These are illustrated in Figure 7. As before, determinacy requires the combination of φ_π and φ_x lie to the Northeast of the dotted boundary. The first new boundary is for the condition $\varphi_\pi < \lambda + \frac{(\beta\lambda-1)\varphi_x}{k}$ which assures that real interest rates will decline as g and inflation rise. (Recall that λ is the autoregressive parameter in the g process, $g_t = \lambda g_{t-1} + \varepsilon_t$, $0 < \lambda < 1$.) The second new boundary is $\varphi_\pi < \lambda + \frac{(\beta\lambda-1)\varphi_x}{k} + \frac{\sigma(1-\lambda)(\beta\lambda-1)}{k}$, which is required in order for

inflation to rise with the g shocks. The shaded area to the left in the Figure 7 represents combinations that provide determinacy, positive impacts of g_t on inflation, and declines in the real interest rates as g_t and inflation both rise. Figure 8 illustrates the impulse responses from DYNARE/MATLAB calculations for a downward movement in real interest rates, in response to a positive shock of g_t , followed by a gradual increase in the real interest rate back to equilibrium.

VIII. Conclusion

The literature has erroneously conflated the Taylor Principle with the NKM determinacy condition. They are two different kettles of fish.

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Figure 1. The Dynamics of the Backward-Looking Model with $\rho = 0.5$

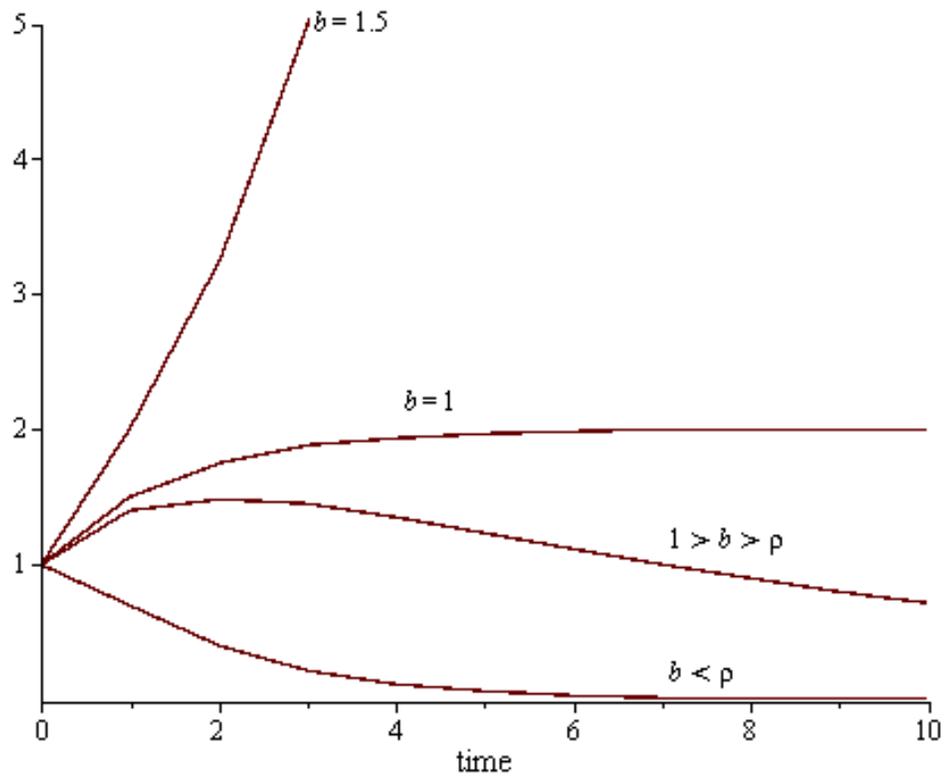


Figure 2. The Dynamics of the Forward-Looking Model with $\rho = 0.5$

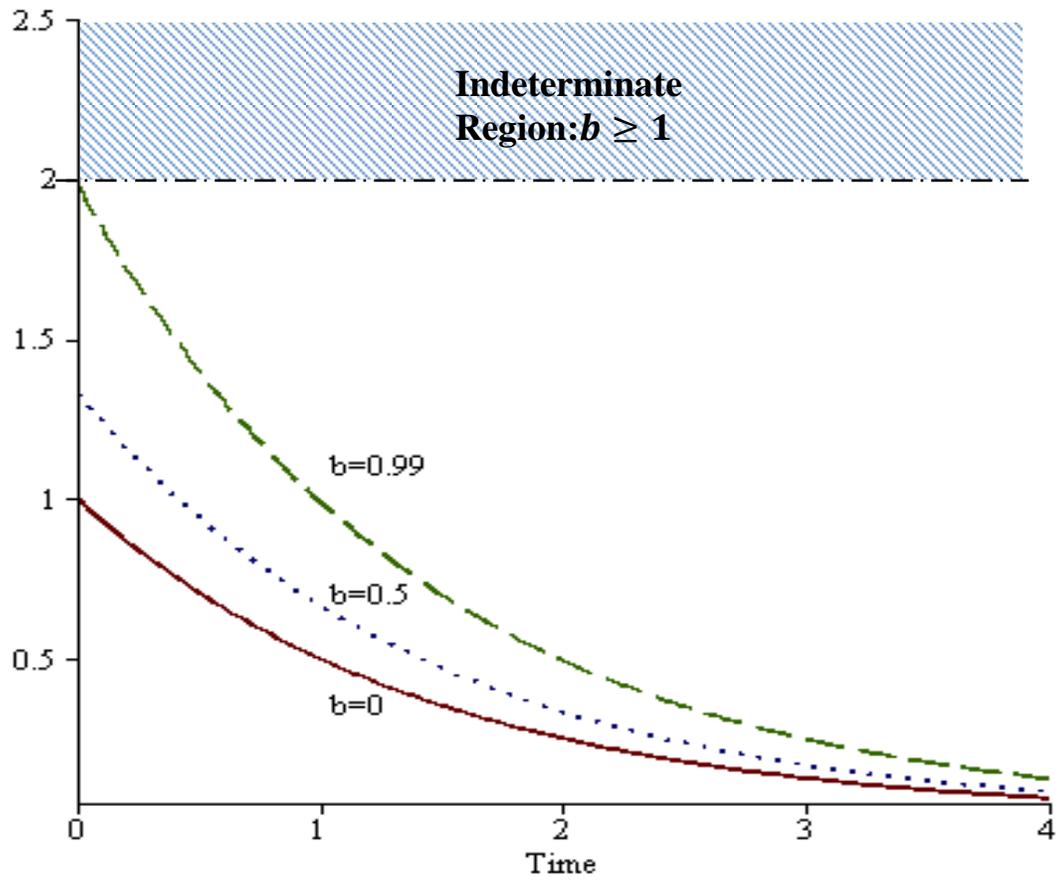
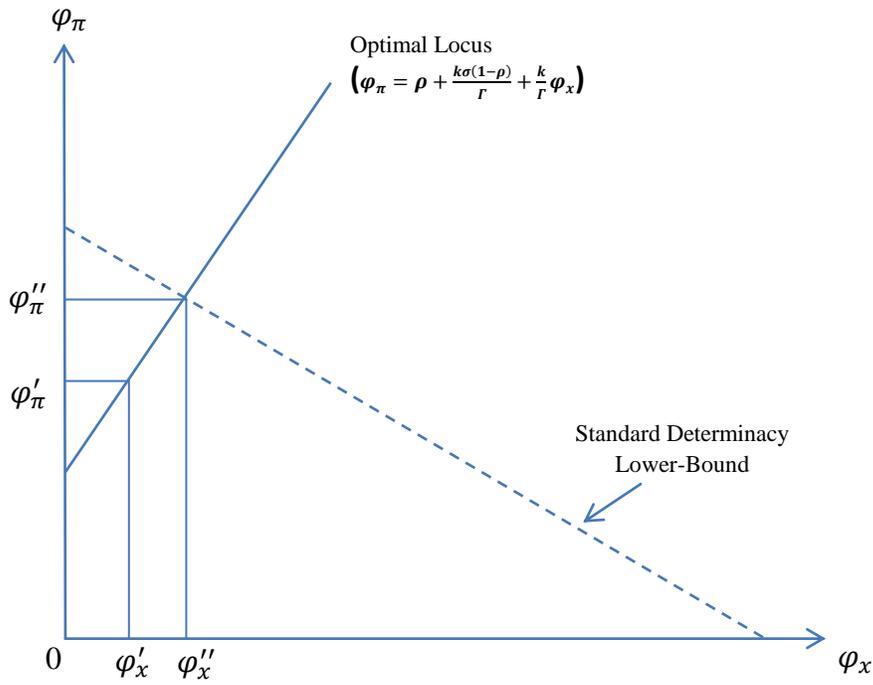
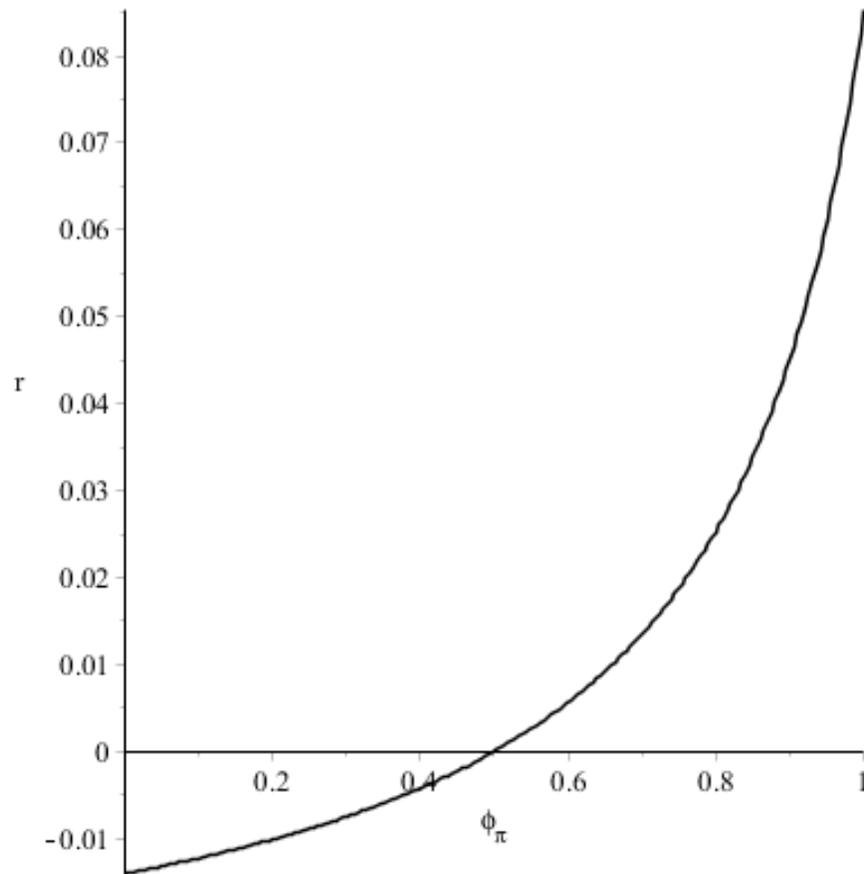


Figure 3. The Optimality Condition and the Determinacy Condition



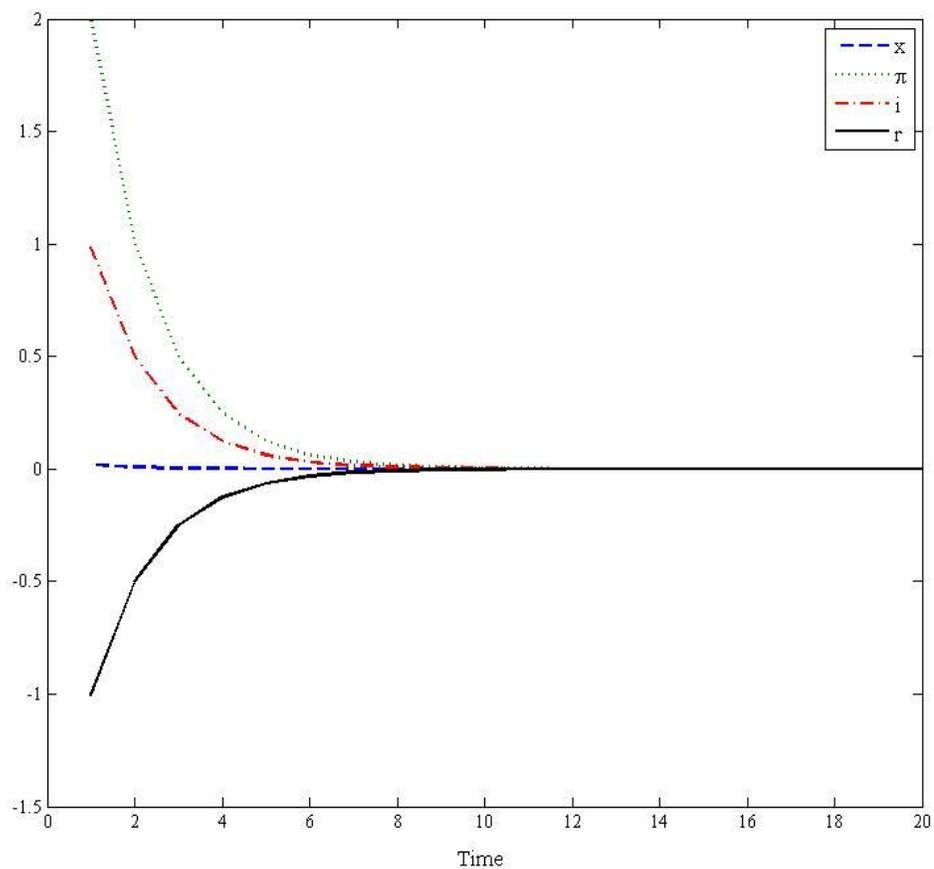
Note: φ_x' and φ_x'' are values that meet optimality/time-consistency and make the coefficient on the future value equal to one. φ_x' and φ_x'' are values that are optimal and meet the standard determinacy condition, $k(\varphi_\pi - 1) + (1 - \beta)\varphi_x > 0$.

Figure 4. Simulated Deviations in the Real Interest Rate Created by a Unit Shock in u_t



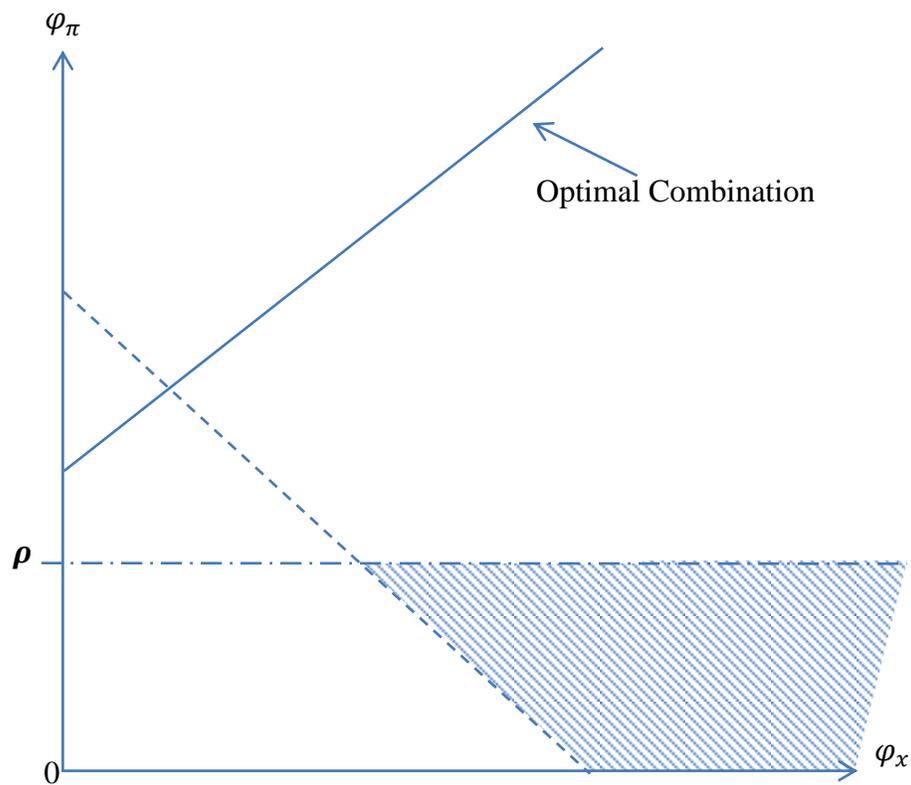
Note: Assumed parameters: $\rho = 0.5$, $\Gamma = 2$, $k = 0.3$, $\sigma = 1$, $\beta = 0.99$. The ϕ_x term is set in each case so that the right side of the determinacy condition (15) is equal to 0.05.

Figure 5. Impulse Responses to u_t : Real Interest Rates Decline with Inflation



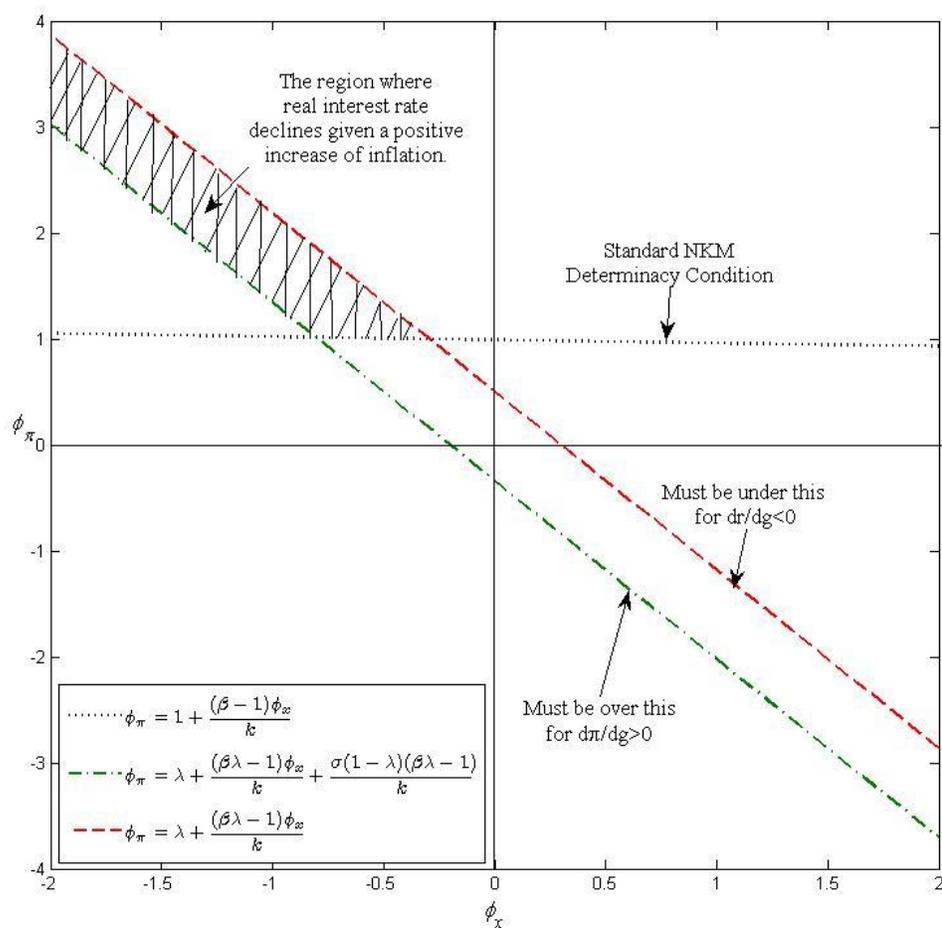
Assumed parameters: $\rho = 0.5$, $\varphi_\pi = 0.3$, $\varphi_x = 22$, $\varphi_g = \sigma = 1$, $k = 0.3$, $\beta = 0.99$

Figure 6. When φ_π is constrained to be less or equal to ρ



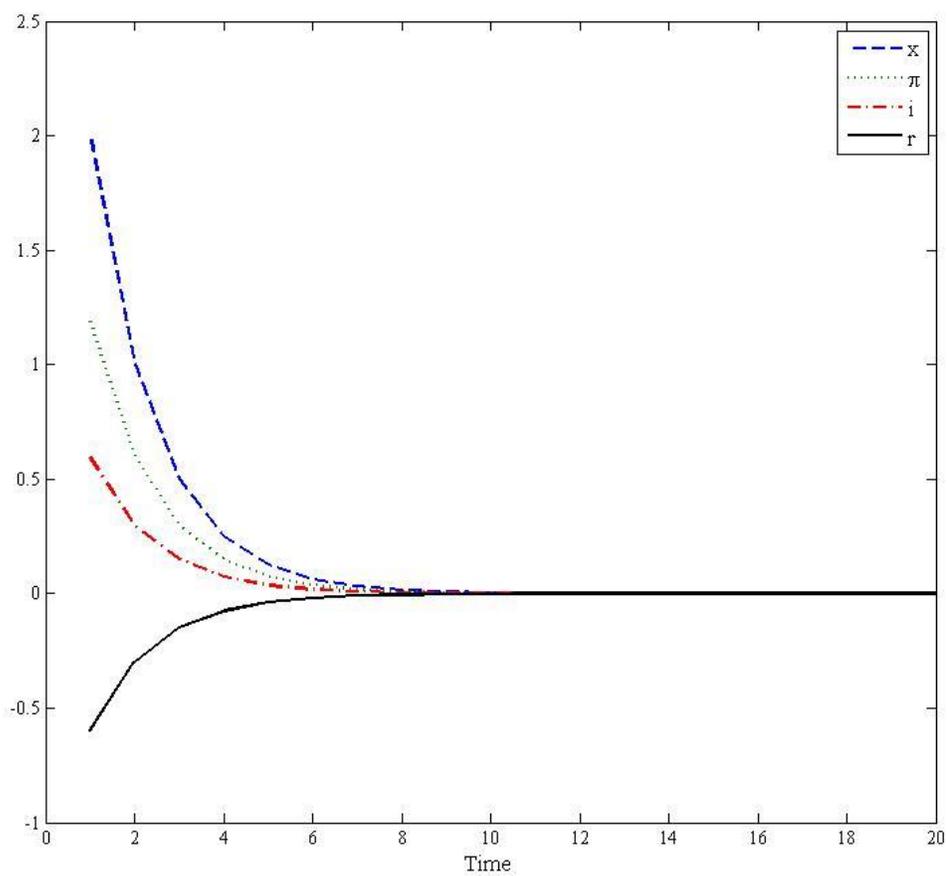
Note: The solid ray denotes the optimal condition $\varphi_\pi = \rho + \frac{k\sigma(1-\rho)}{\Gamma} + \frac{k}{\Gamma}\varphi_x$. The dotted line indicates the boundaries of the determinacy condition $k(\varphi_\pi - 1) + \varphi_x(1 - \beta) > 0$. The intercept of solid ray and the vertical axis must be greater than ρ .

Figure 7. Regions Where the Real Interest Rates Declines when Inflation Increases Upon Arrival of a g Shock



* Shaded area is where demand shocks increase expected inflation which leads to negatively real interest rates effect. Assumed parameters: $\lambda = 0.5$, $\varphi_\pi = 1.02$, $\varphi_x = -0.31$, $\varphi_g = 1$, $k = 0.3$, $\beta = 0.99$, $\sigma = 1$.

Figure 8. Impulse Responses to g_t : Real Interest Rates Decline with Inflation



Assumed parameters: $\lambda = 0.5$, $\varphi_\pi = 1.02$, $\varphi_x = -0.31$, $\varphi_g = 1$, $k = 0.3$, $\beta = 0.99$, $\sigma = 1$

Appendix A

To demonstrate the optimized model of Clarida, Gali and Gertler (1999) has a less “stringent” determinacy condition than the standard determinacy one: first, solve the model for π_t and x_t as function of their two expected values, $E_t[\pi_{t+1}]$ and $E_t[x_{t+1}]$. Then convert these into a two-equation, univariate model using the first order condition, $x_t = -\frac{k}{\Gamma}\pi_t$ and $E_t[x_{t+1}] = -\frac{k}{\Gamma}E_t[\pi_{t+1}]$; also constrain the φ 's to follow the optimal locus (17) ($\varphi_\pi = \rho + \frac{k\sigma(1-\rho)}{\Gamma} + \frac{k}{\Gamma}\varphi_x$).

Taking the univariate equations above for π_t and x_t as functions of $E_t[\pi_{t+1}]$ and $E_t[x_{t+1}]$, respectively, find a value for φ_x that makes the coefficient on the expected future values just equal to unity in each equation (the expression will be the same in both equations). This value of φ_x' shown in Figure 3 is:

$$(A1) \quad \varphi_x' = \frac{(\rho-1)k\Gamma+(2-\rho)k^2\sigma+(1-\beta)\sigma\Gamma}{-k^2-\Gamma+\beta\Gamma}.$$

This establishes the “borderline” φ_x' and the associated φ_π' which will meet the optimality condition and above which make the coefficients on the respective expected values less than unity.

Second, find the intersection of the optimality condition (17) $\varphi_\pi = \rho + \frac{k\sigma(1-\rho)}{\Gamma} + \frac{k}{\Gamma}\varphi_x$ and the lower boundary of the determinacy condition (15) expressed as $k(\varphi_\pi - 1) + (1 - \beta)\varphi_x = 0$.

The intersection solution for is φ_x'' is:

$$(A2) \quad \varphi_x'' = \frac{(\rho-1)(k\Gamma-k^2\sigma)}{-k-\Gamma+\beta\Gamma}.$$

Values of φ_π and φ_x must exceed φ'_x and its associated φ''_π in Figure 3 in order to meet the standard determinacy condition.

Third, subtract (A2) from (A1):

$$(A3) \quad \varphi''_x - \varphi'_x = \frac{-k^2\sigma - (1-\beta)\sigma\Gamma}{-k^2 - (1-\beta)\Gamma}.$$

The value of (A3) is positive given the fact that both numerator and denominator are negative.

The borderline combinations for φ_π and φ_x in the standard case are larger, respectively, than for the optimized model case.

Appendix B

To derive the condition for which the real interest rate will move in the opposite direction of inflation while the model still satisfies the NKM determinacy condition: first, consider the case where inflation is driven exclusively by u_t as in (18). Noting that $\pi_t = \frac{E_t[\pi_{t+1}]}{\rho}$ write the real interest rate $r_t = i_t - E_t[\pi_{t+1}] = \varphi_\pi\pi_t + \varphi_x x_t - E_t[\pi_{t+1}] = \left(\frac{\varphi_\pi}{\rho} - 1\right)E_t[\pi_{t+1}] + \varphi_x x_t$.

Substitute this last expression into (10) and, noting that $E_t[x_{t+1}] = \rho x_t$, derive (18). The NKM determinacy condition will be met with $\varphi_\pi < \rho$ provided that φ_x is large enough.

To obtain conditions where inflation is exclusively driven by g_t , note that $E_t[x_{t+1}] = \lambda x_t$ and $E_t[\pi_{t+1}] = \lambda\pi_t$ and that the real interest rate is proportional to expected inflation:

$$(B1) \quad r_t = \left\{ \varphi_\pi - \left(\frac{(\beta\lambda - 1)\varphi_x}{k} + \lambda \right) \right\} E_t[\pi_{t+1}].$$

One boundary condition used in Figure 7 is that the real interest rate declines in expected inflation; this requires $\varphi_\pi < \left(\frac{(\beta\lambda-1)\varphi_x}{k} + \lambda\right)$. Next, substitute (B1) and (10) into (11) to derive an expression for inflation (and $E_t[x_{t+1}]$) that is a function of g_t , $\pi_t = \left\{ \varphi_\pi - \left[\left(\lambda + \frac{\sigma(1-\lambda)(\beta\lambda-1)}{k} \right) + \frac{(\beta\lambda-1)\varphi_x}{k} \right] \right\} g_t$. For the coefficient on g_t to be positive, this requires the another boundary condition that $\varphi_\pi > \left[\left(\lambda + \frac{\sigma(1-\lambda)(\beta\lambda-1)}{k} \right) + \frac{(\beta\lambda-1)\varphi_x}{k} \right]$ which also is illustrated in Figure 7. Finally, the NKM determinacy condition (15) boundary as illustrated in Figure 7 can be written $\varphi_\pi > 1 - \frac{(1-\beta)}{\kappa} \varphi_x$.

Appendix C

Treatment of the Taylor Principle vs. NKM Determinacy Condition in Recent Literature

Three Levels of Conflation		
Level	Authors	Descriptions
1. Clear-cut statements that “real interest rate rises with inflation” and that condition leads to determinacy in NKM	Woodford (2001, p.233), Walsh (2010, p.342), Gali (2008, pp.78-79), *Cochrane (2011, pp. 572-573, p.583)	Walsh (2010, p.342): “a policy that raised the nominal interest rate when inflation rose, and raised i_t enough to increase the real interest rate so that the output gap fell, would be sufficient to ensure a unique equilibrium.” Gali (2008, pp. 78-79): “the equilibrium will be unique under interest rate rule whenever φ_x and φ_π are sufficiently large enough to guarantee that the real rate eventually rises in the face of an increase in inflation.”
2. The NKM’s determinacy condition simply <u>defined</u> as the “long-run” or “generalized” Taylor Principle. No direct statement that the real interest rate must rise in inflation in order to achieve determinacy, but this is implied.	Ascari & Ropele (2009, p. 1566), Oistein Roisland (2003, pp. 149-150), Llosa & Tuesta (2008, p.1034, p.1042), Llosa & Tuesta (2009, p.1883), Duffy & Xiao (2011, p.975), Davig & Leeper (2007, p.607, p.612), Coibion & Gorodnichenko (2009, p.9, p.11)	Misleading, in that an implication that in the “long run,” or “eventually,” the real rate must rise in order to imply determinacy
3. Model is constructed so that the model will exhibit Taylor Principle <u>and</u> determinacy provide a certain condition holds (for example, $\varphi_\pi > 1$).	Kurozumi & Zandweghe (2008, p.1492), Kurozumi & Zandweghe (2011, p.1026), Duffy & Xiao (2011, p.990), Bullard & Singh (2008, p. S39), Bullard & Schaling (2009, p.1591, p.1609)	Misleading, in that it suggests that somehow the fact that the Taylor Principle holds is responsible for determinacy, when in fact it is by coincidence that both Taylor Principle and determinacy occur

*Cochrane’s paper, unlike others in this table, criticizes the NKM on the grounds that its mechanism toward arriving at determinate equilibrium is ill-conceived. However, his interpretation of the NKM position on this issue is basically the same as the others in Level 1 of this table.

¹ This paper does not analyze “sunspot” solutions. These are rational solutions that reflect certain arbitrary and “non-fundamental” disturbances to expectations which produce self-fulfilling and non-explosive solutions when the mathematical determinacy conditions are not met. When the determinacy conditions are met, the effect is to rule out (in some cases “cancel out”) the effects of the sunspots. Our focus will be on whether meeting the determinacy conditions has dynamic implications apart from ruling out sunspots and/or being instantly explosive. We will argue that determinacy conditions do not have such implications for the baseline New Keynesian Model, notwithstanding widespread claims they do.

² When $n \rightarrow \infty$, give $b < 1$, equation (2) converges to $\tilde{x}_{t+\infty} = \frac{a}{1-b}$.

³ $E_t[u_{t+1}] = b\rho u_t$, $E_{t+1}[u_{t+2}] = b^2\rho^2 u_t$ and $E_{t+n-1}[u_{t+n}] = b^n\rho^n u_t$.

⁴ Cochrane (2011) rewrites the model to put the future dependent variable on the left hand side, which is algebraically sound but interprets it erroneously as a backward model (in his equation (5)). Under this interpretation, future values depend on current values, and the value of φ , which is our b , he claims, suggest that other solutions than the single, stable one will “gradually” explode under restrictions of the equation. This is not valid. There is only one solution, according to our above interpretation, and we will have the same path for dependent value regardless how we write the equation. The fallacy appears to be treating the initial value of the dependent variable as independent of the shocks and future expected values.

⁵ This does not seem to be a particularly important omission, since the Fed is presumably free to increase φ_x as needed to maintain $1 + \frac{2(1+\beta\varphi_x)}{\alpha\beta} > \varphi_\pi$ so as to prevent Taylor “instability.”

⁶ Woodford (2001) proposes this as a necessary parameter to reach optimality in the baseline NKM with Taylor rule.

⁷ This is the first order condition of discretion (time-consistent) solutions of CGG (1999), page 1672.

⁸ We emphasize that this is not the only set of assumptions that can lead to the real interest rate moving in the opposite direction of inflation. For example, Thurston (2012) notes conditions in this model that produce a constant nominal interest rate and a negative co-movement of the real interest rate and inflation.